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Approximation of Time of Ballistic Entry

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Nomenclature

V = magnitude of the velocity vector, fps
γ = attitude of the velocity vector measured positively upward from the local horizontal (for entry trajectories, γ is always negative), deg
D = drag deceleration, $C_D A\ \rho V^2/2m$, ft/sec^2
g = gravitational acceleration, ft/sec^2
r = earth-centered radius vector, ft
ρ = density of the atmosphere, lb/ft^3
β = atmospheric decay coefficient, ft^{-1}
m = mass of the missile, lb
h = altitude, ft
W = weight of the missile, lb (used only in Fig. 1)

Introduction

MANY investigators have studied the segment of the entry trajectory in which the drag deceleration is large compared to the gravitational acceleration. One of the "classical" treatments under this restrictive condition is the one given by Allen and Eggers,[1] which is adequate for determining maximum deceleration and heating effects in the region of peak deceleration. It describes the air drag effects along a straight-line trajectory, since gravity is completely neglected. Moe[2] considers the gravity term in the equations of motion and by a transformation of variables determines an approximate trajectory representing the surface range as an integral solution. (As a special case, when $g = 0$, Moe's solution yields the Allen and Eggers solution.)

Figure 1 shows entry altitude and velocity characteristics for ballistic missiles with constant ballistic coefficients of 1000 and 1500 psf. The region in which gravity can be neglected, as in the Allen and Eggers approximation, is the portion of the trajectory below 90,000-ft alt, i.e., during the last 10 sec prior to impact. Neither Moe's nor Allen and Eggers' solution describes the effect of the air drag upon the entry time. This paper presents a useful method for approximating entry time, based upon the solution describing the exponential decay of the angular momentum on a zero-lift trajectory.

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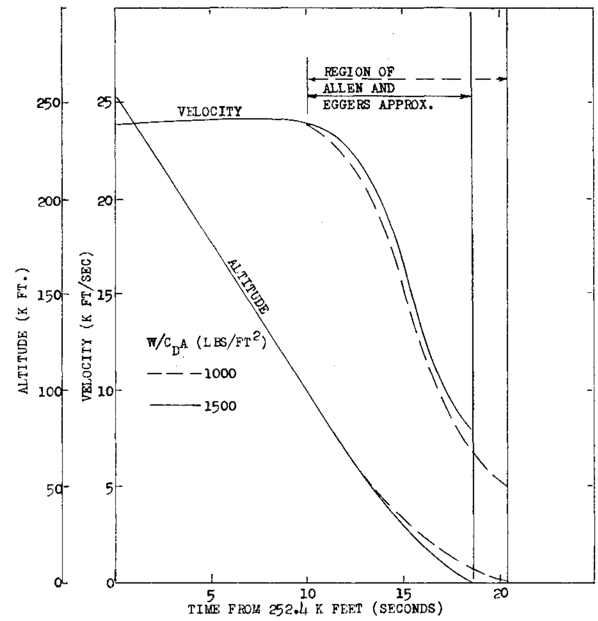

Fig. 1 Altitude and velocity vs time for constant $W/C_D A$ (ballistic range: 5000 naut miles; entry angle: −40°).

Assumptions

The basic assumptions are that $W/C_D A$ is constant and that the equations of motion can be expressed with the lift coefficients set equal to zero. The latter is reasonable for ballistic entry trajectories in which the arrow effect stabilizes the principle axis along the direction of the velocity vector, but it negates the admissibility of cross winds. In general, it can be assumed that ordinary winds and the windage caused by the rotating earth are small in comparison to the laminar flow about the re-entry vehicle caused by the shock wave front.

The atmosphere below 250,000 ft is represented by an exponential fit of the known atmospheric data.† The gravity vector is assumed to be represented by a simple inverse square field in the radius vector; neglecting the oblateness effect of the earth introduces insignificant errors.

A last assumption is that the local attitude angle of the missile's velocity vector with respect to its local horizontal is a slowly varying function and can be replaced by its mean value. For calculation purposes, the initial condition value is chosen instead of the mean value; this replacement introduces a significant error only for altitudes below 25,000 ft.

Decay of the Angular Momentum

In the case of a nonlifting trajectory, the equations of motion in the direction of the velocity vector and normal to the velocity vector are

$$\dot{V} = -g \sin\gamma - D \tag{1}$$

$$V\dot{\gamma} = (-g + V^2/r)\cos\gamma \tag{2}$$

Upon transforming Eq. (1) to the γ derivative, i.e.,

$$\dot{V} = \dot{\gamma}\, dV/d\gamma$$

where

$$\dot{\gamma} = [-(g/V) + (V/r)]\cos\gamma.$$

Eq. (1) becomes

$$\frac{1}{V}\frac{dV}{d\gamma} = \tan\gamma + \frac{D}{g\cos\gamma} + \frac{V}{gr}\frac{dV}{d\gamma} \tag{3}$$

† For some applications, a better curve fit can be obtained by dividing the atmosphere into spherical concentric shells, and curve fitting the exponential model atmosphere to each layer of atmosphere with a continuous transition at the boundaries of the concentric shells.

Table 1 Error of time-predictor equation, $E(t) = T_{IBM} - T_{calc}$[a]

T_{IBM}, sec	$W/C_DA = 1000$ h kft	$E(t)$, sec	$W/C_DA = 1500$ h, kft	$E(t)$, sec
0	252.4	...	252.4	...
3	206.3	−0.005	...	...
6	160.0	−0.003	...	...
9	113.8	−0.001	...	...
12	68.7	+0.04	...	...
15	31.3	+0.15	...	...
18	9.7	+0.64	3.4	+0.19
18.4	...	...	0.0	+1.06
21	0.1	+1.98	...	...

[a] Based on $\beta^{-1} = 23{,}650$ ft; $\rho_{SL} = 0.085$ lb/ft³.

Upon multiplying through by $d\gamma$,

$$\frac{dV}{V} = \tan\gamma d\gamma + \frac{D\dot{\gamma}\, dt}{g\cos\gamma} + \frac{V\dot{V}}{gr}\, dt \tag{4}$$

$$= \tan\gamma\, d\gamma - \frac{D}{V}\, dt - \frac{V\sin\gamma}{r}\, dt \tag{5}$$

$$= \tan\gamma\, d\gamma - \frac{D}{V}\, dt - \frac{dr}{r} \tag{6}$$

Equation (6) is integrable and yields the desired result describing the exponential decay of the angular momentum which forms the basis of this paper.

Integrating Eq. (6) for a constant C_DA/m yields

$$\ln(r\, V\cos\gamma/r_0V_0\cos\gamma_0) = -M/2m \tag{7}$$

where the zero subscript indicates the initial conditions at time $t = 0$, and

$$M = M(t) = \int_{t_0}^{t} C_DA\rho\, V\, dt \tag{8}$$

is the air mass swept out by the effective missile cross-sectional area C_DA, where $\rho = \rho_{SL}\exp(-\beta h)$. Thus, the angular momentum is

$$P = P_0\exp(-M/2m) \tag{9}$$

Time Predictor

Equation (9) can be equivalently stated as

$$d\,(r^2)/dt = 2P_0\exp(-M/2m)\tan\gamma \tag{10}$$

Equation (8) can be written in the form

$$M(h,\gamma) = \int_{h_0}^{h} C_DA\rho_{SL}\exp(-\beta h)\,\frac{dh}{\sin\gamma} \tag{11}$$

With the assumptions that C_DA/m = constant and that $\sin\gamma$ varies slowly throughout the trajectory,

$$M/m = 2a[\exp(-\beta h) - \exp(-\beta h_0)] \tag{12}$$

where

$$a = -C_DA\rho_{SL}/2\beta m\sin\gamma$$

Since γ is negative for entry, the approximated constant a is positive. Then Eq. (10) can be written

$$P_0\tan\gamma\, dt = r\, dr\exp(-ae^{-\beta h_0})\exp(ae^{-\beta h}) \tag{13}$$

Since

$$\exp(ae^{-\beta h}) = \sum_{n=0}^{\infty}\frac{(ae^{-\beta h})^n}{n!}$$

the right-hand side of Eq. (13) can be integrated termwise between the limits of h_0 and h, and the mean value theorem can be applied to the left-hand side. For application to ICBM re-entry trajectories, the mean value of $\tan\gamma$ can be chosen as the initial condition without introducing a significant error. Thus, setting the initial time $T_0 = 0$, the time-predictor equation becomes

$$Tr_0V_0|\sin\gamma_0| = \exp[-ae^{-\beta h_0}]\left[\frac{r_0^2 - r^2}{2} + \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{a^ne^{-n\beta h}}{n\cdot n!}\left(r + \frac{1}{n\beta}\right) - \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{a^ne^{-n\beta h_0}}{n\cdot n!}\left(r_0 + \frac{1}{n\beta}\right)\right] \tag{14}$$

An examination of the convergence of Eq. (14) allows truncation after the first few terms, yielding a practical real-time solution to the time-predictor problem. For a given accuracy, the longest computation involves the summation of the series

$$\sum_{n=1}^{\infty}\frac{a^ne^{-n\beta h}}{n\cdot n!}\left(r + \frac{1}{n\beta}\right)$$

at the lower altitudes. For a computational accuracy of $10^{-4}\,r$, this series requires seven or eight terms for altitudes of 6000 ft or less. The exponential decay constant β and the density constant ρ_{SL} must be selected on the basis of the initial altitude h_0 for a best curve fit.

The equation was programed on a G-15 computer using the initial conditions obtained from an output of a set of IBM programs, which numerically integrated the equations of motion for constant W/C_DA on a 0.1-sec interval. The error of the time-predictor equation is negligible for $h > 50$ kft (Table 1). The initial extremely small errors can be attributed to the slight variations in the initial conditions caused by the five-digit input and calculation accuracy of the G-15 program.

In general, it can be concluded that the approximate solution for the time predictor is applicable to time prediction within a bounded region of a re-entry trajectory. The atmospheric constants ρ_{SL} and β should be selected in terms of the region of application for a best fit time prediction.

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Correlation for Maximum Pressure of "Hi-Lo" Igniter

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The "Hi-Lo" igniter for solid propellant rockets is essentially a monovent rocket that burns pellets of propellant or pyrotechnic material.[1] Analytical prediction of the maximum pressure developed in the igniter by the pellet combustion products is complicated by the transient, short-term nature of the igniter pressurization. For example, a typical pressure-time curve is bell shaped with a zero pressure-to-

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